

Conformal Form of Pseudo-Riemannian Metrics by Normal Coordinate Transformations

A. C. V. V. de Siqueira *
Departamento de Educação
Universidade Federal Rural de Pernambuco
52.171-900, Recife, PE, Brazil.

Abstract

In this paper we extend the Cartan's approach of Riemannian normal coordinates and show that all n -dimensional pseudo-Riemannian metrics are conformal to a flat manifold, when, in normal coordinates, they are well-behaved in the origin and in its neighborhood. We show that for this condition all n -dimensional pseudo-Riemannian metrics can be embedded in a hyper-cone of an $n+2$ -dimensional flat manifold. Based on the above conditions we show that each n -dimensional pseudo-Riemannian manifold is conformal to a n -dimensional manifold of constant curvature. As a consequence of geometry, without postulates, we obtain the classical and the quantum angular momenta of a particle.

* E-mail: acvvs@ded.ufrpe.br

1 Introduction

Conformal spaces are very important in geometry and physics. Researchers pay special attention to them and there are several important results based on or associated with conformal geometry [1], [2]. In this paper we present, in detail, results of Cartan, [3], [4], [5], and make a simple extension that implies a new consequence: all n-dimensional pseudo-Riemannian metrics are conformal to a flat manifold, when, in normal coordinates, they are well-behaved in the origin and in its neighborhood.

This paper is organized as follows. In Sec.2 we present normal coordinates and elements of differential geometry. In Sec.3 we continue the geometric approach. In Sec.4 we show that all well-behaved n-dimensional pseudo-Riemannian metrics in origin and in its neighborhood, in normal coordinates, are conformal to a n-dimensional flat manifold and to a n-dimensional manifold of constant curvature. This result is used in the Cartan's solution for a space of constant curvature. In Sec.5 we present more differential geometry by introducing normal tensors to build the Cartan's solution for a general pseudo-Riemannian metric. In Sec.6 we make an embedding of all n-dimensional pseudo-Riemannian metrics that obey previously presented conditions into a hyper-cone of a flat n+2-dimensional space. In Sec.7, we make an embedding of all n-dimensional pseudo-Riemannian manifold of constant curvature in a n+1-dimensional flat manifold, obtaining, without postulates, the quantum angular momentum operator of a particle as a consequence of geometry.

2 Normal Coordinates

In this section we briefly present normal coordinates and review some elements of differential geometry for an n-dimensional pseudo-Riemannian manifold, [3], [4], [5].

Let us consider the line element

$$ds^2 = G_{\Lambda\Pi} du^\Lambda du^\Pi, \quad (2.1)$$

with

$$G_{\Lambda\Pi} = E_{\Lambda}^{(\mathbf{A})} E_{\Pi}^{(\mathbf{B})} \eta_{(\mathbf{A})(\mathbf{B})}, \quad (2.2)$$

where $\eta_{(\mathbf{A})(\mathbf{B})}$ and $E_{\Lambda}^{(\mathbf{A})}$ are flat metric and vielbein components respectively. We choose each $\eta_{(\mathbf{A})(\mathbf{B})}$ as plus or minus Kronecker's delta function. Let us give the 1-form $\omega^{(\mathbf{A})}$ by

$$\omega^{(\mathbf{A})} = du^{\Lambda} E_{\Lambda}^{(\mathbf{A})}. \quad (2.3)$$

We now define Riemannian normal coordinates by

$$u^{\Lambda} = v^{\Lambda} t, \quad (2.4)$$

then

$$du^{\Lambda} = v^{\Lambda} dt + t dv^{\Lambda}. \quad (2.5)$$

Substituting in (2.3)

$$\omega^{(\mathbf{A})} = t dv^{\Lambda} E_{\Lambda}^{(\mathbf{A})} + dt v^{\Lambda} E_{\Lambda}^{(\mathbf{A})}. \quad (2.6)$$

Let us define

$$z^{(\mathbf{A})} = v^{\Lambda} E_{\Lambda}^{(\mathbf{A})}, \quad (2.7)$$

so that

$$\omega^{(\mathbf{A})} = dt z^{(\mathbf{A})} + t dz^{(\mathbf{A})} + t E^{\Pi(\mathbf{A})} \frac{\partial E_{\Pi(\mathbf{B})}}{\partial z^{(\mathbf{C})}} z^{(\mathbf{B})} dz^{(\mathbf{C})}. \quad (2.8)$$

We now make

$$A^{(\mathbf{A})(\mathbf{B})(\mathbf{C})} = t E^{\Pi(\mathbf{A})} \frac{\partial E_{\Pi(\mathbf{B})}}{\partial z^{(\mathbf{C})}}, \quad (2.9)$$

then

$$\varpi^{(\mathbf{A})} = t dz^{(\mathbf{A})} + A^{(\mathbf{A})(\mathbf{B})(\mathbf{C})} z^{(\mathbf{B})} dz^{(\mathbf{C})}, \quad (2.10)$$

with

$$\omega^{(\mathbf{A})} = dt z^{(\mathbf{A})} + \varpi^{(\mathbf{A})}. \quad (2.11)$$

We have at $t = 0$

$$A^{(\mathbf{A})(\mathbf{B})(\mathbf{C})}(t = 0, z^{(\mathbf{D})}) = 0, \quad (2.12)$$

$$\varpi^{(\mathbf{A})}(t = 0, z^{(\mathbf{D})}) = 0, \quad (2.13)$$

and

$$\omega^{(\mathbf{A})}(t = 0, z^{(\mathbf{D})}) = dt z^{(\mathbf{A})}. \quad (2.14)$$

We conclude that $\omega^{(\mathbf{A})}$ is the one-form associated to the normal coordinate u^Λ , $z^{(\mathbf{A})}$ is associated to the local coordinate v^Λ of a local basis, and $\varpi^{(\mathbf{A})}$ is the one-form associated to the one-form $dz^{(\mathbf{A})}$.

Consider, at a $n+1$ -manifold, a coordinate system given by $(t, z^{(\mathbf{A})})$. For each value of t we have a hyper-surface, where $dt = 0$ on each of them. We are interested in the hyper-surface with $t = 1$. On this hyper-surface we verify the following equality

$$\omega^{(\mathbf{A})}(t = 1, z) = \varpi^{(\mathbf{A})}(t = 1, z). \quad (2.15)$$

The equality (2.15) is true on all hyper-surface $t = \text{constant}$.

Consider the following expression in a vielbein basis

$$d\omega^{(\mathbf{A})} = -\omega_{(\mathbf{B})}^{(\mathbf{A})} \wedge \omega^{(\mathbf{B})}. \quad (2.16)$$

The expression is invariant by coordinate transformations.

Consider now the map Φ , between two manifolds M and N , and consider two subsets, U of M and V of N . Then,

$$\Phi : U \longrightarrow V. \quad (2.17)$$

Define now pull-back as follows, [4],

$$\Phi^* : F^p(V) \longrightarrow F^p(U), \quad (2.18)$$

so that Φ^* sends p -forms into p -forms.

It is well known that the exterior derivative commutes with pull-back, so that

$$\Phi^*(d\omega_{(\mathbf{B})}^{(\mathbf{A})}) = d\Phi^*(\omega_{(\mathbf{B})}^{(\mathbf{A})}), \quad (2.19)$$

and

$$\Phi^*(d\omega^{(\mathbf{A})}) = d\Phi^*(\omega^{(\mathbf{A})}). \quad (2.20)$$

We also have

$$\Phi^*(\omega_{(\mathbf{B})}^{(\mathbf{A})} \wedge \omega^{(\mathbf{B})}) = \Phi^*(\omega_{(\mathbf{B})}^{(\mathbf{A})}) \wedge \Phi^*(\omega^{(\mathbf{B})}). \quad (2.21)$$

The equation (2.11) can be seen as pull-back,

$$\Phi^*(\omega^{(\mathbf{A})}) = dtz^{(\mathbf{A})} + \varpi^{(\mathbf{A})}. \quad (2.22)$$

It can be shown, by a simple calculation that

$$\Phi^*(\omega_{(\mathbf{B})}^{(\mathbf{A})}) = \varpi_{(\mathbf{B})}^{(\mathbf{A})}. \quad (2.23)$$

We note that $dt = 0$, for $\varpi^{(\mathbf{A})}$ and for $\varpi_{(\mathbf{B})}^{(\mathbf{A})}$.

By the exterior derivative of (2.22), we obtain

$$\begin{aligned} d(\Phi^*(\omega^{(\mathbf{A})})) &= d(dt z^{(\mathbf{A})} + \varpi^{(\mathbf{A})}) = dz^{(\mathbf{A})} \wedge (dt) \\ &\quad + dt \wedge \frac{\partial(\varpi^{(\mathbf{A})})}{\partial(t)} \end{aligned} \quad (2.24)$$

+ terms not involving dt .

Making a pull-back of (2.16) and using (2.21) we have

$$\Phi^*(d\omega^{(\mathbf{A})}) = \Phi^*(-\omega_{(\mathbf{B})}^{(\mathbf{A})} \wedge \omega^{(\mathbf{B})}) = -\Phi^*(\omega_{(\mathbf{B})}^{(\mathbf{A})}) \wedge \Phi^*(\omega^{(\mathbf{B})}). \quad (2.25)$$

Using (2.20), (2.23), (2.24) and (2.25) we have

$$\frac{\partial(\varpi^{(\mathbf{A})})}{\partial(t)} = dz^{(\mathbf{A})} + \varpi_{(\mathbf{B})}^{(\mathbf{A})} z^{(\mathbf{D})}. \quad (2.26)$$

We can, by a similar procedure to (2.19), and using the Cartan's second structure equation, obtain the following result

$$\frac{\partial(\varpi_{(\mathbf{A})(\mathbf{B})}^{(\mathbf{A})})}{\partial(t)} = R_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} z^{(\mathbf{C})} \varpi^{(\mathbf{A})}. \quad (2.27)$$

Making a new partial derivative of (2.26), two partial derivatives of (2.10), comparing the results and using (2.27) we have the following equation

$$\frac{\partial^2(A_{(\mathbf{A})(\mathbf{C})(\mathbf{D})})}{\partial(t^2)} = tz^{(\mathbf{B})} R_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + z^{(\mathbf{L})} z^{(\mathbf{M})} R_{(\mathbf{A})(\mathbf{L})(\mathbf{M})(\mathbf{N})} A_{(\mathbf{P})(\mathbf{C})(\mathbf{D})} \eta^{(\mathbf{N})(\mathbf{P})}. \quad (2.28)$$

Rewriting (2.28), with the indices (C) and (D) permuted, we obtain the following result

$$\frac{\partial^2(A_{(\mathbf{A})(\mathbf{D})(\mathbf{C})})}{\partial(t^2)} = tz^{(\mathbf{B})} R_{(\mathbf{A})(\mathbf{B})(\mathbf{D})(\mathbf{C})} + z^{(\mathbf{L})} z^{(\mathbf{M})} R_{(\mathbf{A})(\mathbf{L})(\mathbf{M})(\mathbf{N})} A_{(\mathbf{P})(\mathbf{D})(\mathbf{C})} \eta^{(\mathbf{N})(\mathbf{P})}. \quad (2.29)$$

Adding (2.28) and (2.29) and using the curvature symmetries we have the following solution

$$A_{(\mathbf{A})(\mathbf{C})(\mathbf{D})} + A_{(\mathbf{A})(\mathbf{D})(\mathbf{C})} = 0, \quad (2.30)$$

that is true for all t .

Then,

$$A_{(\mathbf{A})(\mathbf{C})(\mathbf{D})} = -A_{(\mathbf{A})(\mathbf{D})(\mathbf{C})}, \quad (2.31)$$

so that, we can rewrite (2.10) as

$$\varpi^{(\mathbf{A})} = t dz^{(\mathbf{A})} + \frac{1}{2} A^{(\mathbf{A})(\mathbf{B})(\mathbf{C})} (z^{(\mathbf{B})} dz^{(\mathbf{C})} - z^{(\mathbf{C})} dz^{(\mathbf{B})}). \quad (2.32)$$

Let us define

$$A_{(\mathbf{A})(\mathbf{C})(\mathbf{D})} = z^{(\mathbf{B})} B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})}. \quad (2.33)$$

The following result is obtained by substituting (2.33) in (2.28),

$$\frac{\partial^2 (B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})})}{\partial (t^2)} = t R_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + z^{(\mathbf{L})} z^{(\mathbf{M})} R_{(\mathbf{A})(\mathbf{B})(\mathbf{L})(\mathbf{N})} B_{(\mathbf{P})(\mathbf{M})(\mathbf{C})(\mathbf{D})} \eta^{(\mathbf{N})(\mathbf{P})}. \quad (2.34)$$

We now rewrite (2.34) as follows

$$\frac{\partial^2 (B_{(\mathbf{B})(\mathbf{A})(\mathbf{C})(\mathbf{D})})}{\partial (t^2)} = t R_{(\mathbf{B})(\mathbf{A})(\mathbf{C})(\mathbf{D})} + z^{(\mathbf{L})} z^{(\mathbf{M})} R_{(\mathbf{B})(\mathbf{A})(\mathbf{L})(\mathbf{N})} B_{(\mathbf{P})(\mathbf{M})(\mathbf{C})(\mathbf{D})} \eta^{(\mathbf{N})(\mathbf{P})}. \quad (2.35)$$

Adding (2.34) and (2.35) and using the curvature symmetries we obtain the solution

$$B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + B_{(\mathbf{B})(\mathbf{A})(\mathbf{C})(\mathbf{D})} = \text{const.}, \quad (2.36)$$

for all t .

We can use (2.12) and (2.33) in (2.36) to obtain

$$B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + B_{(\mathbf{B})(\mathbf{A})(\mathbf{C})(\mathbf{D})} = 0. \quad (2.37)$$

In the following, for future use, we present the line element on the hypersurface

$$ds'^2 = \eta_{(\mathbf{A})(\mathbf{B})} \varpi^{(\mathbf{A})} \varpi^{(\mathbf{B})}. \quad (2.38)$$

From (2.31), (2.33) and (2.37) we conclude that $B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})}$ has the same symmetries of the Riemann curvature tensor

$$B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} = -B_{(\mathbf{B})(\mathbf{A})(\mathbf{C})(\mathbf{D})} = -B_{(\mathbf{A})(\mathbf{B})(\mathbf{D})(\mathbf{C})}. \quad (2.39)$$

Using (2.31) and (2.37) we have

$$\begin{aligned}
A_{(\mathbf{A})(\mathbf{C})(\mathbf{D})} dz^{(\mathbf{A})} z^{(\mathbf{C})} dz^{(\mathbf{D})} = \\
+ \frac{1}{4} B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} \cdot \\
\cdot (z^{(\mathbf{B})} dz^{(\mathbf{A})} - z^{(\mathbf{A})} dz^{(\mathbf{B})}) \cdot \\
\cdot (z^{(\mathbf{C})} dz^{(\mathbf{D})} - z^{(\mathbf{D})} dz^{(\mathbf{C})}).
\end{aligned} \tag{2.40}$$

Now we can construct the line element of the hyper-surface. By direct use of (2.32) and (2.40) we have

$$\begin{aligned}
ds'^2 = t^2 \eta_{(\mathbf{A})(\mathbf{B})} dz^{(\mathbf{A})} dz^{(\mathbf{B})} + \\
+ \frac{1}{2} \left\{ \frac{1}{2} t \epsilon_{(\mathbf{B})} B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + \right. \\
\left. + \eta^{(\mathbf{M})(\mathbf{N})} A_{(\mathbf{M})(\mathbf{B})(\mathbf{A})} A_{(\mathbf{N})(\mathbf{C})(\mathbf{D})} \right\} \cdot \\
\cdot (z^{(\mathbf{B})} dz^{(\mathbf{A})} - z^{(\mathbf{A})} dz^{(\mathbf{B})}) (z^{(\mathbf{C})} dz^{(\mathbf{D})} - z^{(\mathbf{D})} dz^{(\mathbf{C})}).
\end{aligned} \tag{2.41}$$

The line elements of the manifold and the hyper-surface are equal at $t = 1$, where $u^\Lambda = v^\Lambda$,

$$ds^2 = ds'^2, \tag{2.42}$$

and

$$\begin{aligned}
ds^2 = \eta_{(\mathbf{A})(\mathbf{B})} dz^{(\mathbf{A})} dz^{(\mathbf{B})} + \\
+ \frac{1}{2} \left\{ \frac{1}{2} \epsilon_{(\mathbf{B})} B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + \right. \\
\left. + \eta^{(\mathbf{M})(\mathbf{N})} A_{(\mathbf{M})(\mathbf{B})(\mathbf{A})} A_{(\mathbf{N})(\mathbf{C})(\mathbf{D})} \right\} \cdot \\
\cdot (z^{(\mathbf{B})} dz^{(\mathbf{A})} - z^{(\mathbf{A})} dz^{(\mathbf{B})}) (z^{(\mathbf{C})} dz^{(\mathbf{D})} - z^{(\mathbf{D})} dz^{(\mathbf{C})}).
\end{aligned} \tag{2.43}$$

Note that (2.43) is not an approximation of (2.1), they are equal.

3 Conformal Form of Riemannian Metrics

Sometimes it is possible to write the metric in a particular form, as follows

$$\begin{aligned}
ds^2 = & \eta_{(\mathbf{a})(\mathbf{b})} dz^{(\mathbf{a})} dz^{(\mathbf{b})} + \\
& + \{ \eta_{(0)(0)} + \frac{1}{2} [\frac{1}{2} \epsilon_{(\mathbf{B})} B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + \\
& + \eta^{(\mathbf{M})(\mathbf{N})} A_{(\mathbf{M})(\mathbf{B})(\mathbf{A})} A_{(\mathbf{N})(\mathbf{C})(\mathbf{D})}] \}. \\
& . (z^{(\mathbf{B})} \frac{dz^{(\mathbf{A})}}{d\tau} - z^{(\mathbf{A})} \frac{dz^{(\mathbf{B})}}{d\tau}) (z^{(\mathbf{C})} \frac{dz^{(\mathbf{D})}}{d\tau} - z^{(\mathbf{D})} \frac{dz^{(\mathbf{C})}}{d\tau}) \} d\tau^2,
\end{aligned} \tag{3.1}$$

where $(a), (b) \neq 0$.

Defining

$$\begin{aligned}
d\rho^2 = & \{ \eta_{(0)(0)} + \frac{1}{2} [\frac{1}{2} \epsilon_{(\mathbf{B})} B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + \\
& + \eta^{(\mathbf{M})(\mathbf{N})} A_{(\mathbf{M})(\mathbf{B})(\mathbf{A})} A_{(\mathbf{N})(\mathbf{C})(\mathbf{D})}] \}. \\
& . (z^{(\mathbf{B})} \frac{dz^{(\mathbf{A})}}{d\tau} - z^{(\mathbf{A})} \frac{dz^{(\mathbf{B})}}{d\tau}) (z^{(\mathbf{C})} \frac{dz^{(\mathbf{D})}}{d\tau} - z^{(\mathbf{D})} \frac{dz^{(\mathbf{C})}}{d\tau}) \} d\tau^2,
\end{aligned} \tag{3.2}$$

(3.3)

then, (3.1) can be rewritten as

$$ds^2 = d\rho^2 + \eta_{(\mathbf{a})(\mathbf{b})} dz^{(\mathbf{a})} dz^{(\mathbf{b})}. \tag{3.4}$$

We now write (2.43) as

$$\begin{aligned}
ds^2 = & \eta_{(\mathbf{A})(\mathbf{B})} dz^{(\mathbf{A})} dz^{(\mathbf{B})} + \\
& + \{ \frac{1}{2} [\frac{1}{2} \epsilon_{(\mathbf{B})} B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + \\
& + \eta^{(\mathbf{M})(\mathbf{N})} A_{(\mathbf{M})(\mathbf{B})(\mathbf{A})} A_{(\mathbf{N})(\mathbf{C})(\mathbf{D})}] \}. \\
& . (z^{(\mathbf{B})} \frac{dz^{(\mathbf{A})}}{ds} - z^{(\mathbf{A})} \frac{dz^{(\mathbf{B})}}{ds}) (z^{(\mathbf{C})} \frac{dz^{(\mathbf{D})}}{ds} - z^{(\mathbf{D})} \frac{dz^{(\mathbf{C})}}{ds}) \} ds^2.
\end{aligned} \tag{3.5}$$

It can also be written in the form

$$\begin{aligned}
& [1 - \frac{1}{2}[\frac{1}{2}\epsilon_{(\mathbf{B})}B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + \\
& + \eta^{(\mathbf{M})(\mathbf{N})}A_{(\mathbf{M})(\mathbf{B})(\mathbf{A})}A_{(\mathbf{N})(\mathbf{C})(\mathbf{D})}]]. \\
& .(z^{(\mathbf{B})}\frac{dz^{(\mathbf{A})}}{ds} - z^{(\mathbf{A})}\frac{dz^{(\mathbf{B})}}{ds})(z^{(\mathbf{C})}\frac{dz^{(\mathbf{D})}}{ds} - z^{(\mathbf{D})}\frac{dz^{(\mathbf{C})}}{ds})]ds^2 \\
& = \eta_{(\mathbf{A})(\mathbf{B})}dz^{(\mathbf{A})}dz^{(\mathbf{B})}.
\end{aligned} \tag{3.6}$$

We now define the function

$$L^{\mathbf{A})(\mathbf{B})} = (z^{(\mathbf{B})}\frac{dz^{(\mathbf{A})}}{ds} - z^{(\mathbf{A})}\frac{dz^{(\mathbf{B})}}{ds}), \tag{3.7}$$

which is the classical angular momentum of a free particle.
The line element (3.6) can assume the following form

$$\begin{aligned}
& \{1 + \frac{1}{2}[\frac{1}{2}(\epsilon_{(\mathbf{B})}B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + \\
& + \eta^{(\mathbf{M})(\mathbf{N})}A_{(\mathbf{M})(\mathbf{B})(\mathbf{A})}A_{(\mathbf{N})(\mathbf{C})(\mathbf{D})}]]. \\
& .(L^{\mathbf{A})(\mathbf{B})}L^{\mathbf{C})(\mathbf{D})})\}ds^2 \\
& = (\eta_{(\mathbf{A})(\mathbf{B})}dz^{(\mathbf{A})}dz^{(\mathbf{B})}).
\end{aligned} \tag{3.8}$$

We now define the function

$$\begin{aligned}
\exp(-2\sigma) = \{1 + \frac{1}{2}[\frac{1}{2}(\epsilon_{(\mathbf{B})}B_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} \\
+ \eta^{(\mathbf{M})(\mathbf{N})}A_{(\mathbf{M})(\mathbf{B})(\mathbf{A})}A_{(\mathbf{N})(\mathbf{C})(\mathbf{D})})). \\
.L^{\mathbf{A})(\mathbf{B})}L^{\mathbf{C})(\mathbf{D})}\},
\end{aligned} \tag{3.9}$$

so that, the line element assumes the form

$$ds^2 = \exp(2\sigma)\eta_{(\mathbf{A})(\mathbf{B})}dz^{(\mathbf{A})}dz^{(\mathbf{B})}. \tag{3.10}$$

When transformations like (3.2) are possible, (3.4) will be a flat metric, with the time changed, and it is equivalent to the original metric. The metric (3.10) is conformal to a flat manifold, and we conclude that all n-dimensional pseudo-Riemannian metrics are conformal to flat manifolds, when, in normal coordinates, the transformations are well-behaved in the origin and in its neighborhood. It is important to pay attention to the fact that a normal transformation and its inverse are well-behaved in the region where geodesics are not mixed. Points where geodesics close or mix are known as conjugate points of Jacobi's fields. Jacobi's fields can be used for this purpose. Although this is an important problem, we do not make other considerations about the regions where (3.4) and (3.10) will be valid.

In the next section we present the Cartan's solution for the case where curvature is constant. For the Cartan's solution to a general metric, more geometric objects, like normal tensors, are necessary. This will be presented in section 5.

4 Cartan's Solution for Constant Curvature

In this section we present the Cartan's solution for the constant curvature. The calculation is very simple and was done in [3], and reproduced in detail in [4]. Our objective in this section is only to place the Cartan's solution in the forms (3.4) and (3.10).

Cartan used the signature $(+, +, +, \dots, +)$ and obtained the following line element

$$ds^2 = \sum_{k=1}^n (\varpi^{\mathbf{k}})^2 = \sum_{k=1}^n (dv^{\mathbf{k}})^2 +$$

$$- \left[\frac{|K|\mathbf{r}^2 - \mathbf{S}^2(\mathbf{r}\sqrt{|\mathbf{K}|}\mathbf{t})}{|K|\mathbf{r}^4} \right] \sum_{i < j} (v^{\mathbf{i}} dv^{\mathbf{j}} - v^{\mathbf{j}} dv^{\mathbf{i}})^2,$$
(4.1)

where for $K > 0$

$$\mathbf{S} = \sin(\sqrt{|\mathbf{K}|}\mathbf{t}),$$
(4.2)

and for $K < 0$

$$\mathbf{S} = \sinh(\sqrt{|\mathbf{K}|}\mathbf{t}).$$
(4.3)

We write (4.1) in the form (3.1)

$$\begin{aligned}
ds^2 &= \sum_{k=1}^n (\varpi^{\mathbf{k}})^2 = \sum_{k=1}^n (dv^{\mathbf{k}})^2 \\
&- \left[\frac{|K|\mathbf{r}^2 - \mathbf{S}^2(\mathbf{r}\sqrt{|\mathbf{K}|\mathbf{t}})}{|\mathbf{K}|\mathbf{r}^4} \right] \sum_{i < j} \left(v^i \frac{dv^j}{d\tau} - v^j \frac{dv^i}{d\tau} \right)^2 d\tau^2.
\end{aligned} \tag{4.4}$$

Consider the following function

$$l^{\mathbf{ij}} = \sum_{i < j} \left(v^i \frac{dv^j}{d\tau} - v^j \frac{dv^i}{d\tau} \right)^2. \tag{4.5}$$

Using (4.5) in (4.4) we obtain

$$ds^2 = \sum_{k=1}^n (\varpi^{\mathbf{k}})^2 = \sum_{k=1}^n (dv^{\mathbf{k}})^2 - \left[\frac{|K|\mathbf{r}^2 - \mathbf{S}^2(\mathbf{r}\sqrt{|\mathbf{K}|\mathbf{t}})}{|K|\mathbf{r}^4} \right] \sum_{i<j} (l^{ij})^2 d\tau^2. \quad (4.6)$$

Sometimes we can suppose that $dv^1 = d\tau$. Then, in this case (4.6) can be written in the form

$$ds^2 = \sum_{k=1}^n (\varpi^{\mathbf{k}})^2 = \sum_{k=2}^n (dv^{\mathbf{k}})^2 + \{1 - \left[\frac{|K|\mathbf{r}^2 - \mathbf{S}^2(\mathbf{r}\sqrt{|\mathbf{K}|\mathbf{t}})}{|K|\mathbf{r}^4} \right] \sum_{i<j} (l^{ij})^2\} d\tau^2. \quad (4.7)$$

Defining,

$$d\rho^2 = \{1 - \left[\frac{|K|\mathbf{r}^2 - \mathbf{S}^2(\mathbf{r}\sqrt{|\mathbf{K}|\mathbf{t}})}{|K|\mathbf{r}^4} \right] \sum_{i<j} (l^{ij})^2\} d\tau^2, \quad (4.8)$$

and using it in (4.6), we obtain

$$ds^2 = d\rho^2 + \sum_{k=2}^n (dv^{\mathbf{k}})^2, \quad (4.9)$$

where (4.9) has the same form as (3.4).

We now write (4.1) in the form (3.10). For this we change (4.1) as follows

$$ds^2 = \sum_{k=1}^n (\varpi^{\mathbf{k}})^2 = \sum_{k=1}^n (dv^{\mathbf{k}})^2 + \left[\frac{|K|\mathbf{r}^2 - \mathbf{S}^2(\mathbf{r}\sqrt{|\mathbf{K}|\mathbf{t}})}{|K|\mathbf{r}^4} \right] \sum_{i<j} \left(v^i \frac{dv^j}{ds} - v^j \frac{dv^i}{ds} \right)^2 ds^2. \quad (4.10)$$

We note that (4.10) has the form of (3.5).

Defining

$$L^{(\mathbf{i})(\mathbf{j})} = (z^{(\mathbf{i})} \frac{dz^{(\mathbf{j})}}{ds} - z^{(\mathbf{j})} \frac{dz^{(\mathbf{i})}}{ds}). \quad (4.11)$$

and replacing (4.11) in (4.10) we obtain

$$\begin{aligned} ds^2 &= \sum_{k=1}^n (\varpi^{\mathbf{k}})^2 = \sum_{k=1}^n (dv^{\mathbf{k}})^2 + \\ &- \left[\frac{|K|\mathbf{r}^2 - \mathbf{S}^2(\mathbf{r}\sqrt{|\mathbf{K}|}\mathbf{t})}{|K|\mathbf{r}^4} \right] \sum_{i < j} (L^{(\mathbf{i})(\mathbf{j})})^2 ds^2, \end{aligned} \quad (4.12)$$

which is equivalent to

$$\{1 + \left[\frac{|K|\mathbf{r}^2 - \mathbf{S}^2(\mathbf{r}\sqrt{|\mathbf{K}|}\mathbf{t})}{|K|\mathbf{r}^4} \right] \sum_{i < j} (L^{(\mathbf{i})(\mathbf{j})})^2\} ds^2 = \sum_{k=1}^n (dv^{\mathbf{k}})^2. \quad (4.13)$$

We now define

$$\exp(-2\sigma) = \{1 + \left[\frac{|K|\mathbf{r}^2 - \mathbf{S}^2(\mathbf{r}\sqrt{|\mathbf{K}|}\mathbf{t})}{|K|\mathbf{r}^4} \right] \sum_{i < j} (L^{(\mathbf{i})(\mathbf{j})})^2\}. \quad (4.14)$$

Substituting in (4.13) we obtain

$$ds^2 = \exp(2\sigma) \sum_{k=1}^n (dv^{\mathbf{k}})^2. \quad (4.15)$$

We could have this section with all equations in a vielbein basis. The results would be the same. This will be made at the end of the next section for the general solution.

We rewrite (4.15) as follows

$$ds^2 = \{1 + \left[\frac{|K|\mathbf{r}^2 - \mathbf{S}^2(\mathbf{r}\sqrt{|\mathbf{K}|}\mathbf{t})}{|K|\mathbf{r}^4} \right] \sum_{i < j} (\eta_{\mathbf{ij}} L^{(\mathbf{i})(\mathbf{j})})^2\}^{-1} dv^{\mathbf{l}} dv^{\mathbf{k}} \eta_{\mathbf{lk}}, \quad (4.16)$$

where $\eta_{\mathbf{jk}}$ is a generic flat metric.

By a coordinate transformation we can put (4.16) in the well known form

$$ds^2 = \left\{1 + \frac{K\Omega^{\mathbf{j}}\Omega^{\mathbf{k}}\eta_{\mathbf{jk}}}{4}\right\}^{-2} d\Omega^{\mathbf{j}} d\Omega^{\mathbf{k}} \eta_{\mathbf{jk}}. \quad (4.17)$$

It is well known that (4.17) is conformal to a flat metric. As (4.16) and (4.17) are equivalent, we conclude that (4.16) is also conformal to a flat metric. Therefore, we conclude that there is a local conformal transformation between (4.16) or (4.17) and (3.10). This is an important result that will be analyzed in section 7.

In the next section we present, in detail, some geometric objects, like normal tensors. This is necessary for the Cartan's solution of a general metric.

5 Normal Tensors

In this section, a Taylor's expansion for the metric tensor components will be built in the origin of normal coordinates. Normal tensors are very important for this. In this paper we use the notation (;) for the covariant derivative.

Consider the line element

$$ds^2 = G_{\Lambda\Pi} du^\Lambda du^\Pi. \quad (5.1)$$

Its expansion in the origin of a normal coordinate has the general form

$$\begin{aligned} ds^2 = G_{\lambda\pi} du^\lambda du^\pi &= G_{\lambda\pi}(0) + \frac{\partial G_{\lambda\pi}}{\partial u^\mu} v^\mu t \\ &+ \frac{1}{2} \frac{\partial^2 G_{\lambda\pi}}{\partial u^\mu \partial u^\nu} v^\mu v^\nu t^2 + \dots, \end{aligned} \quad (5.2)$$

where the derivatives are calculated at $u^\pi = 0$.

Some results will be found in [6], [7], but, in general, they are not simple. Our results are simpler because they are more specific.

Consider the covariant derivative of $G_{\lambda\pi}$ at a normal coordinate.

For a pseudo-Riemannian space we have

$$G_{\lambda\pi};_\mu = 0. \quad (5.3)$$

From (5.3) we obtain

$$\frac{\partial G_{\lambda\pi}}{\partial u^\mu} = C_{\mu\lambda}^\rho G_{\rho\pi} + C_{\mu\pi}^\rho G_{\lambda\rho}, \quad (5.4)$$

where

$$C_{\mu\lambda}^\rho(0) = 0, \quad (5.5)$$

and

$$\frac{\partial G_{\lambda\pi}}{\partial u^\mu}(0) = 0, \quad (5.6)$$

in origin.

In the limit $u = 0$, the partial derivatives of (5.4) supply all derivative terms for the expansion (5.2). Each partial derivative of $C_{\mu\lambda}^\rho$, calculated in the origin, is a new tensor. These new tensors are called normal tensors. We designate the following representation for them,

$$D_{\mu\lambda\alpha\beta\dots\gamma}^\rho = \frac{\partial^n C_{\mu\lambda}^\rho}{\partial u^\alpha \partial u^\beta \dots \partial u^\gamma}(0). \quad (5.7)$$

We conclude from (5.7) that normal tensors are symmetric at the first pair of inferior indices and also have a complete symmetry among other inferior indices.

It is simple to show that

$$S(D_{\mu\lambda\alpha\beta\dots\gamma}^\rho) = 0, \quad (5.8)$$

where S designates the sum of different normal tensor components. With (5.4), (5.5), (5.6), (5.7) and (5.8) we can calculate all terms of the expansion (5.2).

Deriving (5.5), calculating the limit, and using (5.7) we have

$$\frac{\partial^2 G_{\lambda\pi}}{\partial u^\mu \partial u^\nu} = G_{\lambda\rho} D_{\mu\pi\nu}^\rho + G_{\pi\rho} D_{\mu\lambda\nu}^\rho. \quad (5.9)$$

There is more than one way of associating the curvature tensor with normal tensors. In the following we present the simplest way we know.

Let us define, in normal coordinates, the following components of the Riemannian curvature tensor

$$R_{\mu\lambda\nu}^\rho = \frac{\partial(C_{\mu\lambda}^\rho)}{\partial u^\nu} - \frac{\partial(C_{\mu\nu}^\rho)}{\partial u^\lambda} + C_{\mu\lambda}^\sigma C_{\sigma\nu}^\rho - C_{\mu\nu}^\sigma C_{\sigma\lambda}^\rho. \quad (5.10)$$

The limit of (5.10) is

$$R_{\mu\lambda\nu}^{\rho} = D_{\mu\lambda\nu}^{\rho} - D_{\mu\nu\lambda}^{\rho}, \quad (5.11)$$

where we have used (5.5) and (5.7).

Using (5.7), (5.8), (5.11) and the symmetries of the Riemannian curvature tensor, we can show that

$$D_{\mu\lambda\nu}^{\rho} = \frac{1}{3}(R_{\mu\lambda\nu}^{\rho} + R_{\lambda\mu\nu}^{\rho}). \quad (5.12)$$

Using (5.9) and (5.12) we obtain

$$\frac{\partial^2 G_{\alpha\beta}}{\partial u^{\gamma} \partial u^{\delta}} u^{\gamma} u^{\delta} = \frac{2}{3} R_{\alpha\gamma\beta\delta} u^{\gamma} u^{\delta}. \quad (5.13)$$

By similar procedure, but tedious calculation, we obtain

$$\frac{\partial^3 G_{\alpha\beta}}{\partial u^{\mu} \partial u^{\nu} \partial u^{\sigma}} u^{\mu} u^{\nu} u^{\sigma} = R_{\alpha\mu\beta\nu;\sigma} u^{\gamma} u^{\delta} u^{\mu} u^{\nu} u^{\sigma}. \quad (5.14)$$

Derivatives of fourth order for metric tensor are easy but very long. We do not present them here.

Now we can conclude the Taylor's expansion of the metric tensor. First we rewrite

$$\begin{aligned} G_{\lambda\pi} &= G_{\lambda\pi}(0) \\ &+ \frac{1}{2} \frac{\partial^2 G_{\lambda\pi}}{\partial u^{\mu} \partial u^{\nu}} v^{\mu} v^{\nu} t^2 \\ &+ \frac{1}{6} \frac{\partial^3 G_{\alpha\beta}}{\partial u^{\mu} \partial u^{\nu} \partial u^{\sigma}} v^{\mu} v^{\nu} v^{\sigma} t^3 + \dots, \end{aligned} \quad (5.15)$$

Now we substitute (5.13) and (5.14) in (5.15) obtaining

$$\begin{aligned} G_{\lambda\pi} du^{\alpha} du^{\beta} &= G_{\alpha\beta}(0) du^{\alpha} du^{\beta} + \\ &+ \frac{1}{3} [R_{\alpha\gamma\beta\delta} t^2 + \\ &+ \frac{1}{2} v^{\sigma} R_{\alpha\mu\beta\nu;\sigma} t^3 + \dots] v^{\gamma} v^{\delta} du^{\alpha} du^{\beta}, \end{aligned} \quad (5.16)$$

Using the symmetries of the curvature tensor we have the following expansion

$$\begin{aligned}
G_{\lambda\pi} du^\alpha du^\beta &= G_{\alpha\beta}(0) du^\alpha du^\beta + \\
&\quad + \frac{1}{12} [R_{\alpha\gamma\beta\delta} t^2 + \\
&\quad + \frac{1}{2} v^\sigma R_{\alpha\gamma\beta\delta;\sigma} t^3 + \dots] [v^\gamma du^\alpha - v^\alpha du^\gamma] [v^\beta du^\delta - v^\delta du^\beta].
\end{aligned}
\tag{5.17}$$

On the hyper-surface $t = 1$ we have $dt = 0$ and

$$\begin{aligned}
G_{\lambda\pi} du^\alpha du^\beta &= G_{\alpha\beta}(0) dv^\alpha dv^\beta + \\
&\quad + \frac{1}{12} [R_{\alpha\gamma\beta\delta} + \\
&\quad + \frac{1}{2} v^\sigma R_{\alpha\gamma\beta\delta;\sigma}] [v^\gamma dv^\alpha - v^\alpha dv^\gamma] [v^\beta dv^\delta - v^\delta dv^\beta].
\end{aligned}
\tag{5.18}$$

which is the same result of Cartan, although, by a different way.

It is always possible to place a flat metric into a diagonal form. This is the case of a metric at the origin of normal coordinates. In this case we have

$$E_\Lambda^{(\mathbf{A})}(0) = \delta_\Lambda^{(\mathbf{A})}. \tag{5.19}$$

We now present the Taylor's expansion of $E_\Lambda^{(\mathbf{A})}$ at the origin of a normal coordinate

$$\begin{aligned}
E_\Lambda^{(\mathbf{A})}(u) &= \delta_\Lambda^{(\mathbf{A})} + \\
&\quad + \frac{\partial(E_\Lambda^{(\mathbf{A})})}{\partial(u^\alpha)} du^\alpha + \dots
\end{aligned}
\tag{5.20}$$

Multiplying (5.18) by the vielbein components and their inverse, using (5.19) and (5.20), we obtain

$$\begin{aligned}
ds^2 = & \eta_{(\mathbf{A})(\mathbf{B})} dz^{(\mathbf{A})} dz^{(\mathbf{B})} + \\
& + \frac{1}{12} [R_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} + \\
& + \frac{1}{2} z^{(\mathbf{M})} R_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D}),(\mathbf{M})}] \cdot \\
& \cdot (z^{(\mathbf{B})} dz^{(\mathbf{A})} - z^{(\mathbf{A})} dz^{(\mathbf{B})}) (z^{(\mathbf{C})} dz^{(\mathbf{D})} - z^{(\mathbf{D})} dz^{(\mathbf{C})}),
\end{aligned} \tag{5.21}$$

where the calculation was made on the hyper-surface $t = 1$ and $dt = 0$. Note that the expansion given by (5.22) is an approximated solution of (2.43). Using a perturbation method, Cartan first solved the equations (2.28) and (2.34) and then placed each solution into (2.43). Following the same procedure used to place (5.18) in the form (5.21), we can place (3.10) as follows

$$\begin{aligned}
ds^2 = & \{1 + \frac{1}{2} [\frac{1}{2} (\epsilon_\beta B_{\alpha\beta\gamma\delta}) \\
& + \eta^{\rho\sigma} A_{\rho\alpha\beta} A_{\sigma\gamma\delta})] \cdot \\
& \cdot L^{\alpha\beta} L^{\gamma\delta} \}^{-1} \eta_{\alpha\beta} d\Omega^\alpha d\Omega^\beta.
\end{aligned} \tag{5.22}$$

We now rewrite (4.17) obtaining

$$ds'^2 = \{1 + \frac{K\Omega^\alpha\Omega^\beta\eta_{\alpha\beta}}{4}\}^{-2} d\Omega^\rho d\Omega^\sigma \eta_{\rho\sigma}. \tag{5.23}$$

Because (5.22) and (5.23) are conformal to a flat manifold, there is a conformal transformation between them, with a conformal factor, $(\exp 2\psi)$. Then

$$g'_{\alpha\beta} = (\exp 2\psi) g_{\alpha\beta}. \tag{5.24}$$

More specifically,

$$\begin{aligned}
& \{1 + \frac{1}{2} [\frac{1}{2} (\epsilon_\beta B_{\alpha\beta\gamma\delta}) + \\
& + \eta^{\rho\sigma} A_{\rho\alpha\beta} A_{\sigma\gamma\delta})] L^{\alpha\beta} L^{\gamma\delta} \} = \\
& = (\exp 2\psi) \{1 + \frac{K\Omega^\alpha\Omega^\beta\eta_{\alpha\beta}}{4}\}^2.
\end{aligned} \tag{5.25}$$

This is an important result with some consequences as we will see. Note that (5.23) is a particular Einstein's space with a constant curvature, where

$$R'_{\alpha\beta} = \frac{R'}{n} g'_{\alpha\beta}, \quad (5.26)$$

and R' is the scalar curvature. Spaces, as the Schwarzschild's, where

$$R_{\alpha\beta} = 0, \quad (5.27)$$

are Einstein's spaces and are not maximally symmetric.

Einstein's spaces with a constant scalar curvature obey homogeneity and isotropy conditions. They are maximally symmetric spaces.

We will be using the following definitions, [8]

$$\Delta_1 \psi = g^{\mu\nu} \psi_{;\mu} \psi_{;\nu}, \quad (5.28)$$

$$\psi_{\mu\nu} = \psi_{;\mu\nu} - \psi_{;\mu} \psi_{;\nu}, \quad (5.29)$$

$$\Delta_2 \psi = g^{\mu\nu} \psi_{;\mu\nu}. \quad (5.30)$$

From (5.24), (5.28), (5.29), and (5.30) we obtain

$$\begin{aligned} \psi_{\mu\nu} = & \frac{1}{(n-2)} (R_{\mu\nu}) \\ & - \frac{1}{(2)(n-1)(n-2)} (g'_{\mu\nu} R' - g_{\mu\nu} R) \\ & - \frac{1}{2} \Delta_1 \psi g_{\mu\nu}. \end{aligned} \quad (5.31)$$

If $g'_{\mu\nu}$ is a metric of an Einstein's space, then (5.31) is simplified to

$$\begin{aligned} \psi_{\mu\nu} = & -\frac{1}{(n-2)} R_{\mu\nu} + \\ & + \left(\frac{1}{(2)(n-1)(n-2)} R + \frac{1}{(2n)(n-1)} R' (\exp 2\psi) - \frac{1}{2} \Delta_1 \psi \right) g_{\mu\nu}. \end{aligned} \quad (5.32)$$

In the region where (2.4) is well-behaved, (5.25) will be possible.

6 Local Embedding of Riemannian Manifolds in Flat Manifolds

In section 3 we presented some considerations about the regions where coordinate transformations are well-defined. We consider that this is the case, where such conditions are satisfied.

Let us rewrite (3.10)

$$ds^2 = \exp(2\sigma)\eta_{(\mathbf{A})(\mathbf{B})}dz^{(\mathbf{A})}dz^{(\mathbf{B})}. \quad (6.1)$$

Defining the following transformation of coordinates, [8],

$$y^{(\mathbf{A})} = \exp(\sigma)z^{(\mathbf{A})}, \quad (6.2)$$

with $(A) = (1, 2, 3, \dots, n)$,

$$y^{n+1} = \exp(\sigma)(\eta_{(\mathbf{A})(\mathbf{B})}z^{(\mathbf{A})}z^{(\mathbf{B})} - \frac{1}{4}), \quad (6.3)$$

and,

$$y^{n+2} = \exp(\sigma)(\eta_{(\mathbf{A})(\mathbf{B})}z^{(\mathbf{A})}z^{(\mathbf{B})} + \frac{1}{4}). \quad (6.4)$$

It is easy to verify that

$$\eta_{\mathbf{AB}}y^{\mathbf{A}}y^{\mathbf{B}} = 0, \quad (6.5)$$

where,

$$\eta_{\mathbf{AB}} = (\eta_{(\mathbf{A})(\mathbf{B})}, \eta_{(\mathbf{n}+1),(\mathbf{n}+1)}, \eta_{(\mathbf{n}+2),(\mathbf{n}+2)}), \quad (6.6)$$

with,

$$\eta_{(\mathbf{n}+1),(\mathbf{n}+1)} = 1, \quad (6.7)$$

and,

$$\eta_{(\mathbf{n}+2),(\mathbf{n}+2)} = -1. \quad (6.8)$$

By a simple calculation we can verify that the line elements are given by

$$ds^2 = \exp(2\sigma)\eta_{(\mathbf{A})(\mathbf{B})}dz^{(\mathbf{A})}dz^{(\mathbf{B})} = \eta_{\mathbf{AB}}dy^{\mathbf{A}}dy^{\mathbf{B}}. \quad (6.9)$$

The equation (6.5) is a hyper-cone in the $(n+2)$ -dimensional flat manifold. The metric (6.1) was embedded in the hyper-cone (6.5) of the $(n+2)$ -dimensional flat manifold. We could present more results about embedding, but we have already reached our objective.

7 Embedding of Manifolds of Constant Curvatures in Flat Manifolds

In this section we embed the n -dimensional manifold (5.23) in a $n+1$ -dimensional flat manifold obtaining, as a geometric result, without postulate, the quantum angular momentum of a particle. Other results will be presented in another section.

We now consider a manifold (5.23) designated by S , embedded in a $n+1$ -dimensional flat manifold. The following constraint is obeyed [9],

$$\eta_{\alpha\beta}x^\alpha x^\beta = K = \epsilon \frac{1}{R^2}, \quad (7.1)$$

where K is the scalar curvature of the n -dimensional manifold (5.23), $\alpha, \beta = (1, 2, \dots, n+1)$ and $\epsilon = (+1, -1)$. For the special case of a n -sphere we use the following notation S^n for (5.23).

It is convenient that we use a local basis $X_\beta = \frac{\partial}{\partial(x^\beta)}$.

We consider a constant vector \mathbf{C} in the $n+1$ -dimensional manifold given by

$$\eta_{\alpha\beta}C^\alpha X^\beta = \eta^{\alpha\beta}C_\alpha X_\beta = C, \quad (7.2)$$

where C^α are constant and \mathbf{N} is a unitary and normal vector to S . We use the symbol $<, >$ for the internal product in the $n+1$ -dimensional flat manifold and $<, >'$ for S .

A constant vector \mathbf{C} can be decomposed into two parts, one in S and the other off S as follows

$$C = \bar{C} + < C, N > N. \quad (7.3)$$

From the definition of \mathbf{N} and (7.1) we obtain

$$N^\alpha = \frac{x^\alpha}{R} \quad (7.4)$$

Let us construct the covariant derivative of \mathbf{C} . We have a local basis and a diagonal and unitary tensor metric, so that the Christoffel symbols are null. Then the covariant derivative of \mathbf{C} in the \mathbf{Y} direction is given by

$$\nabla_Y C = 0. \quad (7.5)$$

It is easy to show that

$$\nabla_Y N = \frac{Y}{R}. \quad (7.6)$$

The Lie derivative of the metric tensor in S is given by [1],

$$L_{\bar{U}} g' = 2\lambda_U g', \quad (7.7)$$

where \mathbf{U} is a constant vector in the flat manifold, and λ_U is the characteristic function. For S, the characteristic function is given by

$$\lambda_U = -\frac{1}{R} \{ < U, N > . \quad (7.8)$$

Substituting (7.8) in (7.7) we have

$$L_{\bar{U}} g' = -2\frac{1}{R} < U, N > g'. \quad (7.9)$$

In the region of S where $< U, N >$ is not null, \bar{U} is a conformal Killing vector and in the region where $< U, N >$ is null, \bar{U} is a Killing vector.

We now consider another constant vector \mathbf{V} in the flat space. The Lie derivative of its projection in S is given by

$$L_{\bar{U}} g' = -2\frac{1}{R} < U, N > g'. \quad (7.10)$$

As we consider a local basis and constant vectors \mathbf{U} and \mathbf{V} , the commutator is given by

$$[U, V] = 0. \quad (7.11)$$

Then,

$$L_{[\bar{U}, \bar{V}]} g' = -2\frac{1}{R} < [U, V], N > g' = 0. \quad (7.12)$$

Regardless \bar{U} and \bar{V} being Killing or conformal Killing vectors, their commutator is a Killing vector. In the following we will show that the commutator $[\bar{U}, \bar{V}]$ is proportional to the quantum angular momentum of a particle. Using (7.3) in the following commutator of elements of the basis, we obtain

$$\begin{aligned} [\bar{U}, \bar{V}] &= \\ &= U^\alpha V^\beta [X_\alpha - \langle X_\alpha, N \rangle N, X_\beta - \langle X_\beta, N \rangle N] = \\ &= U^\alpha V^\beta [\bar{X}_\alpha, \bar{X}_\beta]. \end{aligned} \quad (7.13)$$

We now calculate the commutator of elements of the basis, by parts. We have by simple calculation

$$\langle X_\alpha, N \rangle N = \frac{1}{R} \eta_{\alpha\beta} x^\beta. \quad (7.14)$$

Substituting (7.14) in (7.13) we obtain

$$\begin{aligned} [\bar{X}_\alpha, \bar{X}_\beta] &= \\ &= [X_\alpha, X_\beta] - [X_\alpha, \frac{1}{R} \eta_{\beta\sigma} x^\sigma N] + [X_\beta, \frac{1}{R} \eta_{\alpha\sigma} x^\sigma N] + \\ &\quad + \frac{1}{R^2} [\eta_{\alpha\sigma} x^\sigma N, \eta_{\beta\sigma} x^\sigma N]. \end{aligned} \quad (7.15)$$

In a local basis we have

$$[X_\alpha, X_\beta] = 0, \quad (7.16)$$

$$[\eta_{\alpha\sigma} x^\sigma N, \eta_{\beta\sigma} x^\sigma N] = 0. \quad (7.17)$$

Substituting in (7.15) we obtain

$$\begin{aligned} [\bar{X}_\alpha, \bar{X}_\beta] &= \\ &= \frac{1}{R^2} (\eta_{\alpha\sigma} x^\sigma \frac{\partial}{\partial(x^\beta)} - \eta_{\beta\sigma} x^\sigma \frac{\partial}{\partial(x^\alpha)}) \\ &= \frac{1}{R^2} (x_\alpha \frac{\partial}{\partial(x^\beta)} - x_\beta \frac{\partial}{\partial(x^\alpha)}) \\ &= -i \frac{1}{\hbar} \frac{1}{R^2} L_{\alpha\beta}. \end{aligned} \quad (7.18)$$

Multiplying $L_{\alpha\beta}$ by a vielbein basis we obtain

$$\begin{aligned} L_{(\mathbf{A})(\mathbf{B})} &= \\ &= (i\hbar)(R^2)R_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})}x^{(\mathbf{D})}\eta^{(\mathbf{C})(\mathbf{M})}\frac{\partial}{\partial(x^{\mathbf{M}})}. \end{aligned} \quad (7.19)$$

where

$$\hat{p}_{(\mathbf{M})} = (i\hbar)\frac{\partial}{\partial(x^{\mathbf{M}})} \quad (7.20)$$

is the quantum momentum operator of a particle, and

$$\begin{aligned} R_{(\mathbf{A})(\mathbf{B})(\mathbf{C})(\mathbf{D})} &= \\ &= \frac{1}{R^2}[\eta_{(\mathbf{A})(\mathbf{D})}\eta_{(\mathbf{B})(\mathbf{C})} - \eta_{(\mathbf{A})(\mathbf{C})}\eta_{(\mathbf{B})(\mathbf{D})}] \end{aligned} \quad (7.21)$$

is the curvature of S in the vielbein basis and $\eta_{(\mathbf{A})(\mathbf{C})}$ is diagonal.

We consider as an important observation that the association between the quantum angular momentum operator and the constant curvature operator is allowed in an orthogonal vielbein basis of a Cartesian coordinate, regardless of having a curved or a flat manifold. We have used the embedding of a n-dimensional manifold S in an n+1-dimensional flat manifold, only to obtain the quantum angular momentum operator of a particle, without postulates. We can rewrite (5.19) as follows

$$\begin{aligned} L_{(\mathbf{A})(\mathbf{B})} &= \\ &= (i\hbar)[\eta_{(\mathbf{A})(\mathbf{D})}\eta_{(\mathbf{B})(\mathbf{C})} - \eta_{(\mathbf{A})(\mathbf{C})}\eta_{(\mathbf{B})(\mathbf{D})}] \cdot \\ &\quad \cdot x^{(\mathbf{D})}\eta^{(\mathbf{C})(\mathbf{M})}\frac{\partial}{\partial(x^{\mathbf{M}})}. \end{aligned} \quad (7.22)$$

Note that the coordinates in (7.18) are in the $n+1$ -dimensional flat manifold and $L_{\alpha\beta} \subset S$, so that $L_{\alpha\beta} = 0$ for α or β equal to $n+1$.

Racah has shown that [10] the Casimir operators of any semisimple Lie group can be constructed from the quantum angular momentum (5.22). Each multiplet of semisimple Lie group can be uniquely characterized by the eigenvalues of the Casimir operators.

Although we have built the quantum angular momentum from classical geometric considerations we can write the usual expression for an eigenstate of Casimir operator as follows

$$\hat{C} | \dots \rangle = C | \dots \rangle. \quad (7.23)$$

In the following we calculate the Lie derivative of the $\mathfrak{so}(p, n-p)$ algebra. For the Lie group $\text{SO}(p, q)$ we choose the signature $(p, q) = (p, n-p) = (-, -, -, \dots, -, +, +, \dots, +)$, with the algebra

$$\begin{aligned} [L_{(\mathbf{A})(\mathbf{B})}, L_{(\mathbf{C})(\mathbf{D})}] = & -i(\eta_{(\mathbf{A})(\mathbf{C})}L_{(\mathbf{B})(\mathbf{D})} + \eta_{(\mathbf{A})(\mathbf{D})}L_{(\mathbf{C})(\mathbf{B})} \\ & + \eta_{(\mathbf{B})(\mathbf{C})}L_{(\mathbf{D})(\mathbf{A})} + \eta_{(\mathbf{B})(\mathbf{D})}L_{(\mathbf{A})(\mathbf{C})}). \end{aligned} \quad (7.24)$$

Considering the Lie derivative

$$\begin{aligned} \mathbf{L}_{[L_{(\mathbf{A})(\mathbf{B})}, L_{(\mathbf{C})(\mathbf{D})}]} g' = \\ = -\frac{1}{R} \langle [X_{(\mathbf{A})}, X_{(\mathbf{B})}], [X_{(\mathbf{C})}, X_{(\mathbf{D})}], N \rangle g' = 0, \end{aligned} \quad (7.25)$$

where, for the orthogonal Cartesian coordinates, the vielbein is given by

$$E_{\Lambda}^{(\mathbf{A})} = \delta_{\Lambda}^{(\mathbf{A})}, \quad (7.26)$$

we have

$$[X_{(\mathbf{A})}, X_{(\mathbf{B})}] = [X_{\alpha}, X_{\beta}] = 0. \quad (7.27)$$

Note that g' in S is form-invariant in relation to the Killing's vector ξ [11] and in relation to the algebra of $SO(p,n-p)$ as well. We conclude that the algebra of $SO(p,n-p)$ is a Killing's object. The same is true for the algebra of the Lie group $SO(n)$, where for $SO(n)$ we could choose the signature $(+, +, +, \dots, +, +)$.

The constraint (7.1) is invariant for many of the classical groups. For these groups it is possible to build operators, from the combination of the quantum angular momentum operators, which are Killing's objects in relation to g' . Therefore, the metric is form-invariant in relation to this algebra. It is interesting to see some of these groups in the Cartan's list of irreducible Riemannian globally symmetric spaces, [5], and in [12].

Note that we start from a normal coordinate transformation. In other words, in the region where the transformation (2.4) is well-behaved, we can build (3.10) and by a conformal transformation we have (5.23) which was essential to obtain the quantum angular momentum operator from geometry.

References

- [1] W.C.Weber and S.I.Goldberg, *Queen's Papers in Pure and Applied Mathematics-No.16* (Queen's University.Kingston.Ontaro,1969).
- [2] *in Lectures in Theoretical PhysicsXIII*, (A.O.Barut and W.E.Brittin, Eds.,Colorado Assoc. Universit, Boulder Colo. 1971); R.Maartem and S.D.Maharaj, *Class.Quantum Grav.* (3,1005,1986); N.V.Mitskievich and J.Horsky, *Class.Quantum Grav.*(13,2603,1996); A.J.Keane and R.K.Barrett, *Class.Quantum Grav.*(17,201,2002); R.Banerjee, *Ann.Phys.* (311,245,2004).
- [3] E.Cartan, **Leçons sur la Geometrie des Espaces De Riemann**, (Gauthier-Villars, Paris,1946).
- [4] M.Spivak, **A Comprehensive Introduction to Diferential Geometry**, Volume two, (Publish or Perish, Inc. 1999).
- [5] S.Helgason, **Differential Geometry and Symmetric Spaces** (Academic Press,1962)
- [6] O.Veblen and T.Y.Thomas, *Trans. Am. Math.Society.*,(vol 25,551,1923).
- [7] L.P.Eisenart, **Non Riemannian Geometry** (Dover Publications,2005)
- [8] L.P.Eisenart, **Riemannian Geometry** (Princeton University Press,1997)
- [9] See reference [1]
- [10] W.Greiner and B.Muller, **Quantum Mechanics Symmetries** (Springer,1994).
- [11] S. Weinberg, **Gravitation andCosmology:Principles and Applications of the General Theory of Relativity** (John Wiley Sons1972)
- [12] R.Gilmore, **Lie Groups,Lie Algebras, and Some of Their Applications** (Dover Publications,Inc,2002)